6.7 Inner Product Spaces

We generalize the notion of inner product from $\mathbb{R}^n$ to a general vector space V:

Definition An **inner product** on a vector space V is a function that, to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, associates a real number $\langle\mathbf{u}, \mathbf{v}\rangle$ and satisfies the following axioms, for all $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and all scalars c :

1. $\langle\mathbf{u}, \mathbf{v}\rangle = \langle\mathbf{v}, \mathbf{u}\rangle$
2. $\langle\mathbf{u}+\mathbf{v}, \mathbf{w}\rangle = \langle\mathbf{u}, \mathbf{w}\rangle + \langle\mathbf{v}, \mathbf{w}\rangle$
3. $\langle c\mathbf{u}, \mathbf{v}\rangle = c\langle\mathbf{u}, \mathbf{v}\rangle$
4. $\langle\mathbf{u}, \mathbf{u}\rangle \geq 0$ and $\langle\mathbf{u}, \mathbf{u}\rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$
   A vector space with an inner product is called an **inner product space**.

Example 1 Fix any two positive numbers-say, 4 and 5-and for vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$, set

$$\langle\mathbf{u}, \mathbf{v}\rangle = 4u_1v_1 + 5u_2v_2 \tag{1}$$

Show that equation (1) defines an inner product.

ANS: For axiom 1, $\langle\vec{u}, \vec{v}\rangle = 4u_1v_1 + 5u_2v_2 = 4v_1u_1 + 5v_2u_2 = \langle\vec{v}, \vec{u}\rangle$

For axiom 2, let $\vec{w} = (w_1, w_2)$, then

$$\begin{aligned}\langle\vec{u}+\vec{v}, \vec{w}\rangle &= 4(u_1+v_1)w_1 + 5(u_2+v_2)w_2 \\ &= 4u_1w_1 + 5u_2w_2 + 4v_1w_2 + 5v_2w_2 \\ &= \langle\vec{u}, w\rangle + \langle\vec{v}, w\rangle\end{aligned}$$

For axiom 3,

$$\begin{aligned}\langle c\vec{u}, \vec{v}\rangle &= 4cu_1v_1 + 5cu_2v_2 \\ &= c(4u_1v_1 + 5u_2v_2) \\ &= c\langle\vec{u}, \vec{v}\rangle\end{aligned}$$

For axiom 4, $\langle\vec{u}, \vec{u}\rangle = 4u_1^2 + 5u_2^2 \geq 0$

and $4u_1^2 + 5u_2^2 = 0$ if and only if $u_1 = u_2 = 0$, i.e. $\vec{u} = \vec{0}$

Thus (1) defines an inner product on $\mathbb{R}^2$.

Lengths, Distances, and Orthogonality

- Let V be an inner product space, with the inner product denoted by $\langle\mathbf{u}, \mathbf{v}\rangle$.
- Define the **length**, or **norm**, of a vector v to be the scalar

$$\|\mathbf{v}\| = \sqrt{\langle\mathbf{v}, \mathbf{v}\rangle}$$

- Equivalently, $\|\mathbf{v}\|^2 = \langle\mathbf{v}, \mathbf{v}\rangle$.
- A **unit vector** is one whose length is 1. The distance between $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} - \mathbf{v}\|$.
- Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle\mathbf{u}, \mathbf{v}\rangle = 0$.

An inner product on $\mathbb{P}_n$

- Let $t_0, \ldots, t_n$ be distinct real numbers. For p and q in $\mathbb{P}_n$, define

$$\langle p, q\rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + \cdots + p(t_n)q(t_n) \tag{2}$$

Inner product Axioms 1-3 are readily checked. For Axiom 4, note that

$$\langle p, p\rangle = [p(t_0)]^2 + [p(t_1)]^2 + \cdots + [p(t_n)]^2 \geq 0$$

Also, $\langle\mathbf{0}, \mathbf{0}\rangle = 0$. If $\langle p, p\rangle = 0$, then p must vanish at $n+1$ points: $t_0, \ldots, t_n$. This is possible only if p is the zero polynomial, because the degree of p is less than $n+1$. Thus (2) defines an inner product on $\mathbb{P}_n$.

Example 2 Consider $\mathbb{P}_2$ with the inner product given by evaluation at $-1, 0$, and 1.

(1) Compute $\langle p, q\rangle$, where $p(t) = 3t - t^2, q(t) = 3 + 2t^2$.

(2) Compute $\|p\|$ and $\|q\|$, for p and q in (1).

(3) Compute the orthogonal projection of q onto the subspace spanned by p, for p and q in (1).

ANS: (1) The inner product is

$$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

So $$\langle 3t - t^2, 3 + 2t^2\rangle = -4\times 5 + 0\times 3 + 2\times 5 = -10$$

(2) $$\|p\| = \sqrt{\langle p, p\rangle}$$

$$\langle p, p\rangle = \langle 3t - t^2, 3t - t^2\rangle$$

$$= (-4)\times(-4) + 0\times 0 + 2\times 2 = 20$$

$$\|p\| = \sqrt{20}$$

$$\langle q, q\rangle = \langle 3+2t^2, 3+2t^2\rangle$$

$$= 5\times5 + 3\times3 + 5\times5$$

$$= 59$$

$$\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{59}$$

(3). The orthogonal projection $\hat{q}$ of q onto the subspace spanned by p is

$$\hat{q} = \frac{\langle q, p\rangle}{\langle p, p\rangle} p = \frac{-10}{20}(3t - t^2)$$

$$\Rightarrow \quad \hat{q} = -\frac{1}{2}(3t - t^2)$$

The Gram-Schmidt Process

The existence of orthogonal bases for finite-dimensional subspaces of an inner product space can be established by the Gram-Schmidt process, just as in $\mathbb{R}^n$.

Example 3 Let V be $\mathbb{P}_4$ with the inner product given by the evaluation at $-2, -1, 0, 1$, and 2, and view $\mathbb{P}_2$ as a subspace of V. Produce an orthogonal basis for $\mathbb{P}_2$ by applying the Gram-Schmidt process to the polynomials $1, t$, and t^2.

ANS: We will use the Gram-Schmidt Process in §6.4 to produce the orthogonal basis.

Notice the inner products depend only on the polynomial values at $-2, -1, 0, 1, 2$. We list them for $1, t, t^2$ as vectors in $\mathbb{R}^5$ for later computation.

polynomials:	1	t	t^2
Their values at $-2,-1,0,1,2$	$\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$	$\begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}$	$\begin{bmatrix}4\\1\\0\\1\\4\end{bmatrix}$

Let $p_0(t) = 1$

Notice that $\langle t, p_0(t) \rangle = -2-1+0+1+2 = 0$

This means t is orthogonal to $p_0(t) = 1$.

So we take $p_1(t) = t$.

Then
$$p_2(t) = t^2 - \frac{\langle t^2, p_1(t)\rangle}{\langle p_1(t), p_1(t)\rangle} p_1(t) - \frac{\langle t^2, p_0(t)\rangle}{\langle p_0(t), p_0(t)\rangle} p_0(t)$$

$$= t^2 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle} t - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} \times 1$$

$$= t^2 - \cancel{\frac{0}{\langle t, t\rangle}} t - \frac{10}{5}$$

$$\Rightarrow p_2(t) = t^2 - 2$$

Thus an orthogonal basis for the the subspace $\mathbb{P}_2$ of $V = \mathbb{P}_4$ is

$$p_0(t) = 1, \quad p_1(t) = t, \quad p_2(t) = t^2 - 2$$

Best Approximation in Inner Product Spaces

Example 4. Let V be $\mathbb{P}_4$ with the inner product in **Example 3**, and let p_0, p_1, and p_2 be the orthogonal basis found in **Example 3** for the subspace $\mathbb{P}_2$.

Find the best approximation to $p(t) = 5 - \frac{1}{2}t^4$ by polynomials in $\mathbb{P}_2$.

ANS: The best approximation to $p(t)$ by polynomials in $\mathbb{P}_2$ is

$$\hat{p} = \text{proj}_{\mathbb{P}_2} p = \frac{\langle p, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p, p_1\rangle}{\langle p_1, p_1\rangle} p_1 + \frac{\langle p, p_2\rangle}{\langle p_2, p_2\rangle} p_2$$

We record the values for p_0, p_1, p_2 and p at $-2, -1, 0, 1, 2$, as the following for later computation.

$$\begin{array}{cccc}
p_0 = 1 & p_1 = t & p_2 = t^2 - 2 & p = 5 - \frac{1}{2}t^4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 \\ -1 \\ -2 \\ -1 \\ 2 \end{bmatrix} & \begin{bmatrix} -3 \\ 9/2 \\ 5 \\ 9/2 \\ -3 \end{bmatrix}
\end{array}$$

So $\langle p, p_0\rangle = -3 + \frac{9}{2} + 5 + \frac{9}{2} - 3 = 8$, $\langle p_0, p_0\rangle = 5$

$\langle p, p_1\rangle = 6 - \frac{9}{2} + \frac{9}{2} - 6 = 0$, $\langle p_1, p_1\rangle = 10$

$\langle p, p_2\rangle = -6 - \frac{9}{2} - 10 - \frac{9}{2} - 6 = -31$, $\langle p_2, p_2\rangle = 14$

So $$\hat{p} = \text{proj}_{\mathbb{P}_2} p = \frac{\langle p, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p, p_1\rangle}{\langle p_1, p_1\rangle} p_1 + \frac{\langle p, p_2\rangle}{\langle p_2, p_2\rangle} p_2$$

$$= \frac{8}{5} p_0 + 0 + \frac{-31}{14} p_2$$

Thus $\hat{p} = \frac{8}{5} - \frac{31}{14}(t^2 - 2)$ is the closest to $p = 5 - \frac{1}{2}t^4$ of all polynomials in $\mathbb{P}_2$.

when the distance between the polynomials is measured at $-2, -1, 0, 1, 2$.

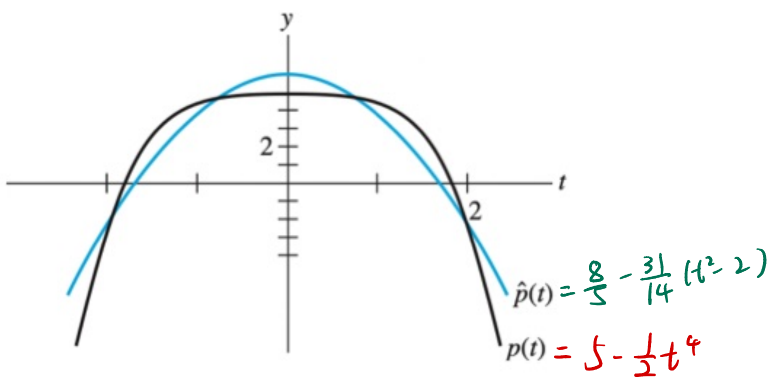

FIGURE 1

An Inner Product for $C[a,b]$ Vector space of all $\overset{\text{real-value}}{\text{continuous}}$ functions defined on $[a, b]$.

For f, g in $C[a,b]$, set

$$\langle f, g\rangle = \int_a^b f(t)g(t)dt \tag{3}$$

Then (3) defines an inner product on $C[a,b]$. Since

- Inner product Axioms 1-3 follow from elementary properties of definite integrals. For Axiom 4, observe that

$$\langle f, f\rangle = \int_a^b [f(t)]^2 dt \geq 0$$

- The function $[f(t)]^2$ is continuous and nonnegative on $[a,b]$. If the definite integral of $[f(t)]^2$ is zero, then $[f(t)]^2$ must be identically zero on $[a,b]$, by a theorem in advanced calculus, in which case f is the zero function.
- Thus $\langle f, f\rangle = 0$ implies that f is the zero function on $[a,b]$. So (3) defines an inner product on $C[a,b]$.

Example 5 Let V be the space $C[-1,1]$ with the inner product

Recall $\int t^n dt = \frac{1}{n+1} t^{n+1} + C$

$$\langle f, g\rangle = \int_{-1}^1 f(t)g(t)dt.$$

Find an orthogonal basis for the subspace spanned by the polynomials $1, t$, and t^2.

ANS: We will use the Gram-Schmidt process in §6.4 to produce the orthogonal basis.

Notice that 1, and t are orthogonal since

$$\langle 1, t\rangle = \int_{-1}^1 t\, dt = \frac{1}{2}t^2\Big|_{-1}^1 = 0$$

So we can take the first two elements of the orthogonal basis to be 1 and t.

By the Gram-Schmidt process, the third basis element can be computed as

$$t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle} t$$

Note

$$\langle t^2, 1\rangle = \int_{-1}^{1} t^2 dt = \frac{1}{3} t^3 \Big|_{-1}^{1} = \frac{2}{3}$$

$$\langle 1, 1\rangle = \int_{-1}^{1} dt = t \Big|_{-1}^{1} = 2$$

$$\langle t^2, t\rangle = \int_{-1}^{1} t^3 dt = \frac{1}{4} t^4 \Big|_{-1}^{1} = 0$$

So $t^2 - \frac{\frac{2}{3}}{2} \times 1 - 0 = t^2 - \frac{1}{3}$

can be the third element of the orthogonal basis. We can scale it to be $3t^2 - 1$

Therefore an orthogonal basis for span $\{1, t, t^2\}$ is $\{1, t, 3t^2 - 1\}$.

Exercise 6. Let $\mathbb{P}_3$ have the inner product given by evaluation at $-3, -1, 1$, and 3.

Let $p_0(t) = 1, p_1(t) = t$, and $p_2(t) = t^2$.
a. Compute the orthogonal projection of p_2 onto the subspace spanned by p_0 and p_1.
b. Find a polynomial q that is orthogonal to p_0 and p_1, such that $\{p_0, p_1, q\}$ is an orthogonal basis for Span $\{p_0, p_1, p_2\}$. Scale the polynomial q so that its vector of values at $(-3, -1, 1, 3)$ is $(1, -1, -1, 1)$.

Solution. The inner product is $\langle p, q\rangle = p(-3)q(-3) + p(-1)q(-1) + p(1)q(1) + p(3)q(3)$.

a. The orthogonal projection $\hat{p}_2$ of p_2 onto the subspace spanned by p_0 and p_1 is
$\hat{p}_2 = \frac{\langle p_2, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle} p_1 = \frac{20}{4}(1) + \frac{0}{20} t = 5.$

b. The vector $q = p_2 - \hat{p}_2 = t^2 - 5$ will be orthogonal to both p_0 and p_1 and $\{p_0, p_1, q\}$ will be an orthogonal basis for $\operatorname{Span}\{p_0, p_1, p_2\}$. The vector of values for q at $(-3, -1, 1, 3)$ is $(4, -4, -4, 4)$, so scaling by $1/4$ yields the new vector $q = (1/4)\left(t^2 - 5\right)$.

Exercise 7. Let $\mathbb{P}_3$ have the inner product as in Exercise 6, with p_0, p_1, and q the polynomials described there. Find the best approximation to $p(t) = t^3$ by polynomials in $\operatorname{Span}\{p_0, p_1, q\}$.

Solution. The best approximation to $p = t^3$ by vectors in $W = \operatorname{Span}\{p_0, p_1, q\}$ will be

$$\hat{p} = \operatorname{proj}_W p = \frac{\langle p, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p, p_1\rangle}{\langle p_1, p_1\rangle} p_1 + \frac{\langle p, q\rangle}{\langle q, q\rangle} q = \frac{0}{4}(1) + \frac{164}{20}(t) + \frac{0}{4}\left(\frac{t^2 - 5}{4}\right) = \frac{41}{5} t.$$